AIAA JOURNAL
Vol. 40, No. 11, November 2002

Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed six manuscript pages and three figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Domain Decomposition for Reduced-Order Modeling of a Flow with Moving Shocks

David J. Lucia* and Paul I. King†
U.S. Air Force Institute of Technology, Wright–Patterson Air Force Base, Ohio 45433
and
Philip S. Beran‡
U.S. Air Force Research Laboratory, Wright–Patterson Air Force Base, Ohio 45433

Introduction

PROPER orthogonal decomposition (POD) is a technique to reduce a high-degree-of-freedom (DOF) flowfield representation to a lower-order system to achieve computational speed. The challenge in the use of POD for high-speed flowfields is capturing moving shock waves. Current techniques[1] that generate POD-based reduced-order models (POD/ROMs) for low-speed flows will not generate a useful ROM for a high-speed case with moving shocks. Excessive data and time are required, and the technique does not track changes in shock location as the boundary conditions or flow parameters change. A new shock-capturing technique is developed to exploit POD for accurately treating moving shock waves. This technique involves decomposition of the solution domain to isolate regions that contain shocks. A reduced-order model for each region is developed independently, and the solution for the entire domain is formed through a linking of the boundaries of each region. This technique is applied to a one-dimensional quasi-steady flowfield for demonstration, although it is extendable to higher dimensional problems requiring either steady or unsteady analysis.[2,3]

POD/ROM and Discontinuities

POD modes constitute an optimal basis[1] for the linear combination of a set of data (snapshots). These data are collected from the time evolution of the flowfield solution or alternatively from a collection of steady-state solutions corresponding to variation in parameters. Initial experimentation showed that POD/ROM could replicate either time-accurate or steady motion of shocks in the flow if all of the shock locations were sampled by the snapshots. Design iteration with POD/ROMs, however, might require a range of shock motion outside the original set of candidate snapshots. Because POD/ROM translates discontinuities from one location to another by linear addition, the POD/ROM would require modes with discontinuities at every possible shock location. Such modes result only when snapshots are obtained for every shock location.

To illustrate these difficulties, and demonstrate the shock-capturing technique, a quasi-steady analysis of POD/ROM for high-speed flows with moving shocks was conducted on an inviscid one-dimensional flowfield in a quasi one-dimensional divergent nozzle (Fig. 1). In a quasi-one-dimensional analysis the area change of the nozzle is replaced by a forcing function in the one-dimensional Euler equations to obtain a solution at the centerline of the nozzle.[4] A numerical solver based on Roe's scheme provided the solution from which various POD/ROMs were generated and compared. Subspace projection and the method of snapshots were employed to obtain POD/ROMs.[5]

Small changes in the ratio of specific heats (γ) moved the steady-state shock location incrementally downstream. A POD/ROM trained with snapshots taken at $\gamma = 1.4$ failed to converge to a steady-state solution for γ below 1.398. Convergence failed because the spatial location of the shock for $\gamma < 1.398$ was downstream of the shock location in the snapshots at $\gamma = 1.4$. Time integration of the POD/ROM failed attempting to form a discontinuity from modes that only supported continuous behavior at this downstream location. POD/ROMs for high-speed flows thus require at least one snapshot for each grid point traversed by a shock wave, and they are not robust to parameter changes in the flowfield that introduce new shock behavior.

Domain Decomposition

Domain decomposition refers to separation of the fluid domain spatially into regions, isolating the region containing the shock wave. For the regions of the flowfield not containing a shock, a POD/ROM is generated as for low-speed flow problems. For the shocked region two alternatives were investigated. One is use of the full-order simulation. Although the shocked region will experience no reduction in the number of DOFs, the nonshocked regions will experience a significant reduction. Another possibility was the use of a reduced-order model for the shocked region of the flow. Because the isolated shock domain is now a small portion of the entire solution domain, snapshot collection is efficient, and the reduced-order model is generated inexpensively.

The domain for the quasi-one-dimensional nozzle problem was decomposed into three regions, as shown in Fig. 2. Section I included the region of nozzle inlet downstream prior to the shock, with supersonic flow throughout. Section II included the exit of section I to just downstream of the shock, thus including a supersonic inlet and a subsonic exit. Section III consisted of the remainder of the nozzle to the exit and contained all subsonic flow. For both cases sections I and III were modeled with POD/ROMs. For section II both full-order and reduced-order modeling were examined.

An implicit non-Galerkin formulation was developed for the POD/ROM domain decomposition (POD/ROM/DD), which required a reordering of the full-order solution vector $\boldsymbol{x}(t)$. Instead

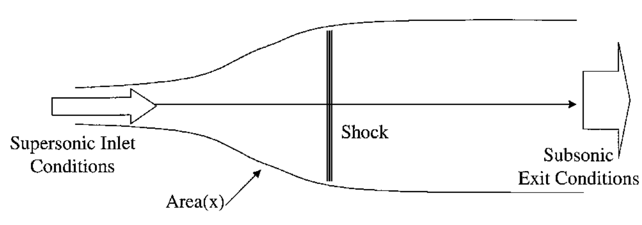

Fig. 1 Quasi-one-dimensional nozzle.

Presented as Paper 2001-2912 at the 15th AIAA Computational Fluid Dynamics Conference, Anaheim, CA, 11–14 June 2001; received 30 August 2001; revision received 20 March 2002; accepted for publication 2 August 2002.

*Doctoral Candidate. Member AIAA.
†Associate Professor. Senior Member AIAA.
‡Senior Research Aerospace Engineer. Associate Fellow AIAA.

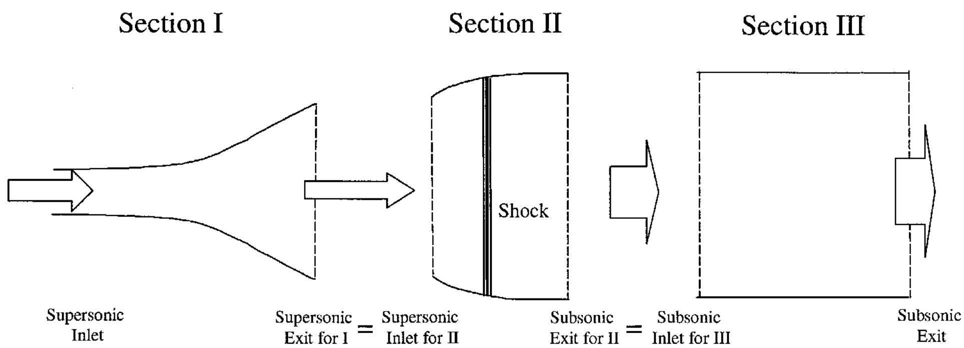

Fig. 2 Domain decomposition for the quasi-one-dimensional nozzle.

of ordering the fluid variable members of $x(t)$ as a distribution across the entire nozzle, $x_{DD}(t)$ stacked the fluid variables across each section [see Eq. (2)]. Each of the three sections was treated as an independent fluid problem governed by x_{S1}, x_{S2}, and x_{S3}. POD produced a reduced-order mapping for each section relating the full-order and reduced-order solutions. Reduced-order variables (denoted with a caret) were $x_{S1} = \Psi_1 \cdot \hat{x}_{S1}$, $x_{S2} = \Psi_2 \cdot \hat{x}_{S2}$, and $x_{S3} = \Psi_3 \cdot \hat{x}_{S3}$. For section II, modeled at full order, $\hat{x}_{S2} = x_{S2}$, and Ψ_2 was an identity mapping. Flow variables at interior boundaries were obtained from the adjacent domain. The full-order flow solver was adjusted to consider each section independently and was denoted $F[x_{DD}(t)]$.

An implicit solver was developed using the chord method to approximate numerically the Jacobian for Newton iterations. The three domains were solved simultaneously as three independent problems with the Newton iterations. With section II modeled with POD/ROM and no domain overlap, a nonphysical discontinuity appeared at the intersection between sections II and III for $\gamma < 1.4$. To enforce smoothness at this intersection with the implicit solver, a constrained optimization technique was developed for use with the reduced-order solver. Other applications of domain decomposition that use constrained optimization have recently emerged in the literature.[6–10]

In the constrained optimization technique sections II and III were allowed to overlap by a few grid points. A functional $\ell(x_{DD})$ was defined such that $d\ell(x_{DD})/dx_{DD} = F(x_{DD})$. Minimizing $\ell(x_{DD})$ was equivalent to solving $F(x_{DD}) = 0$ or finding the steady-state solution to the Euler equations. A constraint was introduced to force the flowfield from both sections to match in the overlapping portion of their respective domains. The constraint included a vector T with the dimensions of x_{DD} such that the dot product of x_{DD} with T tended to zero when the flow variables for the overlapping sections were equal. T was formed by placing a 1 in each fluid variable location corresponding to the overlap in section II, a -1 in each overlapping location in section III, and zeros everywhere else in T. The dot product of T with x_{DD} resulted in cancellation of the fluid variables when the overlapping portion of the solution domain was equivalent. For overlapping fluid variables that are not identical, the dot product is a small scalar residual.

Lagrange constrained optimization minimized $\ell(x_{DD})$ subject to the constraint $x_{DD} \cdot T = 0$ through the use of a Lagrange multiplier λ, introduced as an additional degree of freedom. Lagrange-constrained optimization[11] modifies $\ell(x_{DD})$ by adding the linear constraint to form the function $Q(y)$:

$$Q(y) = \ell(x_{DD}) + \lambda x_{DD} \cdot T \tag{1}$$

The solution vector is augmented to include λ:

$$y = \begin{bmatrix} x_{S1} \\ x_{S2} \\ x_{S3} \\ \lambda \end{bmatrix} = \begin{bmatrix} x_{DD} \\ \lambda \end{bmatrix} \tag{2}$$

Q is minimized for the values of x_{DD} and λ if

$$G(y) = \frac{dQ(y)}{dy} = \begin{bmatrix} F(x_{DD}) + \lambda T \\ x_{DD}^T \cdot T \end{bmatrix} = [0] \tag{3}$$

The reduced-order mapping was defined to include the Lagrange multiplier:

$$y = \begin{bmatrix} \Psi_1 & [0] & [0] & 0 \\ [0] & \Psi_2 & [0] & 0 \\ [0] & [0] & \Psi_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_{S1} \\ \hat{x}_{S2} \\ \hat{x}_{S3} \\ \lambda \end{bmatrix} \tag{4}$$

which was rewritten as

$$y = \Psi_\lambda \hat{y} \tag{5}$$

Newton iterations were used to solve the reduced-order system for $\hat{y}$. The flowfied was obtained by expanding $\hat{x}_{S1}$, $\hat{x}_{S2}$, and $\hat{x}_{S3}$ with reduced-order mappings for each section, after which λ was discarded. The reduced-order Jacobian was obtained from the full-order Jacobian:

$$\frac{dG(y)}{dy} = \begin{bmatrix} \dfrac{dF(x_{DD})}{dx_{DD}} & T \\ T^T & 0 \end{bmatrix} \tag{6}$$

$$\frac{d\hat{G}(\hat{y})}{d\hat{y}} = \left(\Psi_\lambda^T \Psi_\lambda\right)^{-1} \Psi_\lambda^T \frac{dG(y)}{dy} \Psi_\lambda \tag{7}$$

Additional implementation details are contained in Lucia et al.[12]

Results

The implicit methodology successfully produced results for the POD/ROM/DD with shock regions modeled at both full order and with POD/ROM. The 250 grid-point full-order nozzle problem was divided into three sections: section I had 117 grid points, section II had 15 grid points plus 2 coincident grid points for a total of 17, and section III had 118 grid points. For the case of section II modeled at full order with 17 grid points, section I used one mode per fluid variable, and section III used one mode for both density and momentum and two modes for energy. The resulting POD/ROM/DD had 58 DOFs, a significant reduction from the original 750. The second case included a POD/ROM for section II generated from 25 snapshots of steady-state flow solutions at values of gamma from 1.4 to 1.35. An additional four grid points were added to section II as overlap with section III. The POD/ROMs for sections I and III were identical to those for the full-order section II case. Good results were obtained using 16 modes per fluid variable in section II. The eigenvalues associated with these modes showed significant energy (order 10^{-3}) in the 16th mode. Energy in the 17th mode was of order 10^{-7}. The resulting POD/ROM/DD had 55 DOFs, 7 for sections I and III and 48 for section II.

Steady-state solutions from both POD/ROM/DD cases were obtained for γ varied from 1.4 to 1.37. The POD/ROMs for sections I and III were generated from snapshots taken using $\gamma = 1.4$. The

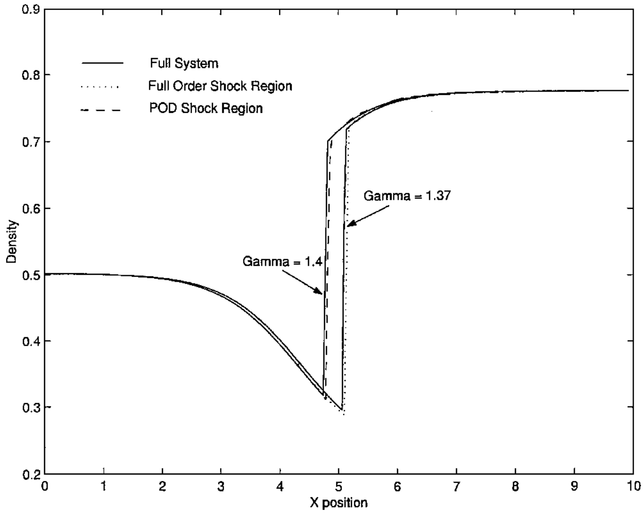

Fig. 3 Shock motion with varying γ.

density results shown in Fig. 3 show that both POD/ROM/DDs tracked the quasi-steady shock motion very closely (within one grid point). In addition, both POD/ROM/DDs accurately solved the flowfield before and after the shock. Density solutions after the shock for both POD/ROM/DDs exhibited a small error growth as gamma was steadily decreased, which occurred because the POD/ROMs for sections I and III were trained at $\gamma = 1.4$. Even with the small error growth, however, the error in density was less than 1%. Errors in density before the shock for both POD/ROM/DDs exhibited a slightly larger error growth (about 2%).

Conclusions

In high-speed flows shock movement can result in the failure of conventional POD/ROM to arrive at a solution. In these cases the POD/ROM will go unstable attempting to form the shock if the shock location is not captured in the snapshots. A new domain decomposition shock-capturing approach was developed to treat moving shocks. The accuracy and order reduction of the domain decomposition approach was demonstrated for quasi-one-dimensional nozzle flow. The nonshocked regions of this flowfield were modeled with POD/ROM trained for $\gamma = 1.4$. The shocked region of the flowfield was modeled both by POD/ROM and by the full-order computational fluid dynamics model adapted for this region. The accuracy of both models was examined for quasi-steady shock motion as γ was varied from 1.4 to 1.37. Both cases produced accurate flowfields and shock motion. Flowfield errors were less than 2%, and the shock movement was tracked within one grid point of the true shock location.

Both methods exhibited similar order reduction.The full-order solution had 750 DOFs, the POD/ROM/DD with a full-order shock region had 58 DOFs, and the POD/ROM/DD with a POD/ROM for the shock region had 55 DOFs. Sixteen modes per fluid variable were required for POD/ROM in the shocked region, resulting in a small-order reduction relative to the full-order shock case. Because of the computational expense of generating snapshots and the large number of modes required, there are no advantages in using POD/ROM for the shocked region for this one-dimensional case. In two- and three-dimensional cases, however, there will be a significant order reduction gained with a POD/ROM for the regions containing shocks.[2,3] The encouraging news from this research is that a small set of DOFs exists that can accurately handle the moving shock case. Future research should focus on efficient methods of solving for these modal coefficients.

References

[1]Holmes, P., Lumley, J., and Berkooz, G., *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*, Cambridge Univ. Press, Cambridge, England, U.K., 1996, pp. 93–107.

[2]Lucia, D. J., "Reduced Order Modeling for High Speed Flows with Moving Shocks," Ph.D. Dissertation, Air Force Inst. of Technology, School of Engineering and Management, Wright–Patterson AFB, OH, Nov. 2001.

[3]Lucia, D. J., Beran, P. S., and King, P. I., "Reduced Order Modeling of an Elastic Panel in Transonic Flow," AIAA Paper 2002-1594, April 2002; also *Journal of Aircraft* (to be published).

[4]Shubin, G., Stephens, A., and Glaz, H., "Steady Shock Tracking and Newton's Method Applied to One-Dimensional Duct Flow," *Journal of Computational Physics*, Vol. 39, 1981, pp. 364–374.

[5]Beran, P., and Silva, W., "Reduced Order Modeling: New Approaches for Computational Physics," AIAA Paper 2001-0853, Jan. 2001.

[6]Lions, J., and Pironneau, O., "Non Overlapping Domain Decomposition for Evolution Operators," *Comptes Rendus de l'Academie des Sciences. Serie I. Mathematique*, Vol. 330, No. 10, 2000, pp. 943–951.

[7]Lions, J., and Pironneau, O., "Overlapping Domain Decomposition for Evolution Operators," *Comptes Rendus de l'Academie des Sciences. Serie I. Mathematique*, Vol. 330, No. 10, 2000, pp. 937–943.

[8]Gunzburger, M. D., and Lee, H. K., "An Optimization-Based Domain Decomposition Method for the Navier–Stokes Equations," *SIAM Journal of Numerical Analysis*, Vol. 37, No. 5, 2000, pp. 1455–1480.

[9]Du, Q., and Gunzburger, M. D., "A Gradent Approach to Optimization-Based Multidisciplinary Simulations and Nonoverlapping Domain Decomposition Algorithms," *SIAM Journal of Numerical Analysis*, Vol. 37, No. 5, 2000, pp. 1513–1541.

[10]Farhat, C., Macedo, A., Lesoinne, M., Roux, F., Magoules, F., and de La Bourdonnaie, A., "Two-Level Domain Decomposition Methods with Lagrange Multipliers for the Fast Iterative Solution of Acoustic Scattering Problems," *Computer Methods in Applied Mechanics and Engineering*, Vol. 184, No. 2, 2000, pp. 213–241.

[11]Strang, G., *Introduction to Applied Mathematics*, Wellesley Cambridge Press, Cambridge, MA, 1986, pp. 96–100.

[12]Lucia, D. J., King, P. I., Beran, P. S., and Oxley, M. E.,"Reduced Order Modeling for a One-Dimensional Nozzle Flow with Moving Shocks," AIAA Paper 2001-2602, June 2001.

H. M. Atassi
Associate Editor

Near-Exact Analytical Solutions of Linear Time-Variant Systems

Anand Natarajan,* Rakesh K. Kapania,† and Daniel J. Inman‡
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061-0203

Introduction

LINEAR time-variant equations, such as the Mathieu–Hill equation, occur in many applications, such as dynamic buckling of structures and wave propagation in periodic media. Periodic variation of parameters in mechanical devices is also common, such as in the meshing of spur gears.[1] One can come across exponential or hyperbolic functions in the coefficients of the differential equation of motion of cables with varying length. Analytical procedures adopted to solve these types of equations involve complex mathematics, even for a one-dimensional problem.[2] Another instance of an exponential variation in system parameters is in structures with an adaptive nature. For example, the active control of the stiffness of vehicle

Received 7 February 2001; presented as Paper 2001-1295 at the AIAA/ASME/ASCE/ASC 42nd Structures, Structural Dynamics, and Materials Conference, Seattle, WA, 16–19 April 2001; revision received 5 December 2001; accepted for publication 10 July 2002.

*Graduate Research Assistant, Department of Aerospace and Ocean Engineering.

†Professor, Department of Aerospace and Ocean Engineering. Associate Fellow AIAA.

‡G. R. Goodson Professor, Department of Mechanical Engineering and Director, Center for Intelligent Material Systems and Structures. Fellow AIAA.